MOREILLON Philippe - Homework 1 ST 2940

Problem 1 : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{A}$ a sub-σ-algebra of $\mathcal{F}$.

To show : if $X = \mathbb{E}[Y|\mathcal{A}]$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$, then $X = Y$ a.s.

$X = Y$ a.s $\Leftrightarrow \mathbb{E}[(X-Y)^2] = 0 \Leftrightarrow \mathbb{E}[X^2] - 2\mathbb{E}[XY] + \mathbb{E}[Y^2] = 0$.

def of conditional expectation

Now $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[Y|\mathcal{A}] \cdot Y] = \mathbb{E}[\mathbb{E}[Y|\mathcal{A}] \cdot \mathbb{E}[Y|\mathcal{A}]] = \mathbb{E}[X^2]$

and $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$ by definition

$\Rightarrow \mathbb{E}[X^2] - 2\mathbb{E}[XY] + \mathbb{E}[Y^2] = \mathbb{E}[X^2] - 2\mathbb{E}[X^2] + \mathbb{E}[X^2] = 0$.

Problem 2 : Let X, Y be jointly distributed such that $Y|X=x \in Bin(n, x)$
$X \in U(0,1)$

Compute the characteristic function of Y and $Cov(X,Y)$.

$$\mathbb{E}[e^{itY}] = \mathbb{E}[\mathbb{E}[e^{itY}|X]] = \int \sum_{k=0}^{n} e^{itk}\, \mathbb{P}(Y=k|X=x)\, f_X(x)\,dx = \int_0^1 \sum_{k=0}^{n} e^{itk} \binom{n}{k} x^k (1-x)^{n-k}\,dx$$

$$= \sum_{k=0}^{n} e^{itk} \binom{n}{k} \int_0^1 x^k (1-x)^{n-k}\,dx \qquad (*)$$

Now let's compute $\int_0^1 x^k(1-x)^{n-k}dx$: $\int_0^1 x^k(1-x)^{n-k}dx = x^k \frac{1}{n-k+1}(1-x)^{n-k+1}\cdot(-1)\Big|_0^1 - \int_0^1 k x^{k-1}\frac{1}{n-k+1}(1-x)^{n-k+1}(-1)\,dx$

$$= \int_0^1 \frac{k}{n-k+1} x^{k-1}(1-x)^{n-(k-1)}dx = \frac{k}{n-(k-1)}\int_0^1 x^{(k-1)}(1-x)^{n-(k-1)}dx \qquad \forall k \geq 1$$

Now $\int_0^1 (1-x)^n dx = \frac{-1}{n+1}(1-x)^{n+1}\Big|_0^1 = \frac{1}{n+1} \Rightarrow \int_0^1 x^k(1-x)^{n-k}dx = \frac{1}{n+1}\cdot\prod_{i=1}^{k}\frac{i}{n-(i-1)} = \frac{k!}{(n+1)\cdot n \cdot \ldots \cdot (n-(k-1))}$

$$= \frac{k!\,(n-k)!}{(n+1)!} = \frac{1}{(n+1)\binom{n}{k}} \Rightarrow (*) = \sum_{k=0}^{n} e^{itk}\cdot\frac{1}{n+1} \underset{\uparrow}{=} \frac{1}{(n+1)}\cdot\frac{1-e^{i(n+1)t}}{1-e^{it}}$$

, i.e $Y \in U(\{0,1,2,\ldots,n\})$

whenever $e^{it} \neq 1$

Let's compute $Cov(X,Y)$: $Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\underbrace{\mathbb{E}[Y|X]}_{=nX \text{ since } Y|X=x \in Bin(n,x)}] = \mathbb{E}[nX^2] = n\mathbb{E}[X^2] = n\int_0^1 x^2dx = n\cdot\frac{1}{3}x^3\Big|_0^1 = \frac{n}{3}$$

$\mathbb{E}[X] = \frac{1}{2}$

$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[nX] = \frac{n}{2}$ $\Rightarrow Cov(X,Y) = \frac{n}{3} - \frac{n}{4} = n\left(\frac{4-3}{12}\right) = \frac{n}{12}$

Problem 3 Let X_1, X_2, X_3 and X_4 be independent and $Exp(\lambda)$ - distributed.

To show : Y_1, Y_2, Y_3, Y_4 are independent, where $Y_1 = X_1 + X_2 + X_3 + X_4$

$$Y_2 = \frac{X_1}{X_1+X_2}$$

$$Y_3 = \frac{X_1+X_2}{X_1+X_2+X_3}$$

$$Y_4 = \frac{X_1+X_2+X_3}{X_1+X_2+X_3+X_4}$$

and determine the joint law of (Y_1, Y_2, Y_3, Y_4).

We have to compute the joint distribution via a change of variables and check that it factorizes.

Fix $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4 \in \mathbb{R}_{\geq 0}$.

$$\mathbb{P}(Y_1 \leq \tilde{y}_1, Y_2 \leq \tilde{y}_2, Y_3 \leq \tilde{y}_3, Y_4 \leq \tilde{y}_4) = \int_0^{\tilde{y}_1}\int_0^{\tilde{y}_2}\int_0^{\tilde{y}_3}\int_0^{\tilde{y}_4} f_{(Y_1,Y_2,Y_3,Y_4)}(y_1,y_2,y_3,y_4)\, dy_1\, dy_2\, dy_3\, dy_4$$

Now we want to find F bijective such that $F(y_1, y_2, y_3, y_4) = (x_1, x_2, x_3, x_4)$

$$\begin{cases} y_1 = x_1+x_2+x_3+x_4 \\ y_2 = \frac{x_1}{x_1+x_2} \\ y_3 = \frac{x_1+x_2}{x_1+x_2+x_3} \\ y_4 = \frac{x_1+x_2+x_3}{x_1+x_2+x_3+x_4} \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 y_2 y_3 y_4 \\ x_2 = y_1(1-y_2)y_3y_4 \\ x_3 = y_1(1-y_3)y_4 \\ x_4 = y_1(1-y_4) \end{cases}$$

$$y_1 \cdot y_2 \cdot y_3 \cdot y_4 = x_1$$

$$y_2 = \frac{x_1}{x_1+x_2} \Leftrightarrow x_2 y_2 = x_1(1-y_2) \Leftrightarrow x_2 = \frac{x_1(1-y_2)}{y_2} = y_1 \cdot (1-y_2) y_3 y_4$$

$$y_3 = \frac{x_1+x_2}{x_1+x_2+x_3} \Leftrightarrow x_3 y_3 = (x_1+x_2)(1-y_3) \Leftrightarrow x_3 = \frac{(x_1+x_2)(1-y_3)}{y_3} = \frac{y_1 y_3 \cdot y_4\, (y_2 + 1 - y_2)(1-y_3)}{y_3}$$

$$= y_1 \cdot (1-y_3) y_4$$

$$y_4 = \frac{x_1+x_2+x_3}{x_1+x_2+x_3+x_4} \Leftrightarrow x_4 y_4 = (x_1+x_2+x_3)(1-y_4) \Leftrightarrow x_4 = \frac{(x_1+x_2+x_3)(1-y_4)}{y_4} = y_1 \cdot (1-y_4)$$

Hence $\mathbb{P}(Y_1 \leq \tilde{y}_1, Y_2 \leq \tilde{y}_2, Y_3 \leq \tilde{y}_3, Y_4 \leq \tilde{y}_4) = \mathbb{P}(F(Y_1,Y_2,Y_3,Y_4) \in F([0,\tilde{y}_1]\times[0,\tilde{y}_2]\times[0,\tilde{y}_3]\times[0,\tilde{y}_4]))$

$= \mathbb{P}((X_1,X_2,X_3,X_4) \in F([0,\tilde{y}_1]\times[0,\tilde{y}_2]\times[0,\tilde{y}_3]\times[0,\tilde{y}_4]))$

by construction of F

$$= \iiiint_{F([0,\tilde{y}_1]\times[0,\tilde{y}_2]\times[0,\tilde{y}_3]\times[0,\tilde{y}_4])} f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) f_{X_4}(x_4)\, dx_1\, dx_2 \ldots dx_4 = \int_0^{\tilde{y}_1}\int_0^{\tilde{y}_2}\int_0^{\tilde{y}_3}\int_0^{\tilde{y}_4} f_{X_1}(y_1y_2y_3y_4) f_{X_2}(y_1(1-y_2)y_3y_4) f_{X_3}(y_1(1-y_3)y_4) f_{X_4}(y_1(1-y_4))\, |\det(J)|\, dy_1\, dy_2\, dy_3\, dy_4$$

$$J = \begin{pmatrix} y_2y_3y_4 & y_1y_3y_4 & y_1y_2y_4 & y_1y_2y_3 \\ (1-y_2)y_3y_4 & -y_1y_3y_4 & y_1(1-y_2)y_4 & y_1(1-y_2)y_3 \\ (1-y_3)y_4 & 0 & -y_1y_4 & y_1(1-y_3) \\ 1-y_4 & 0 & 0 & -y_1 \end{pmatrix}$$

$$= \int_0^{\tilde y_1}\int_0^{\tilde y_2}\int_0^{\tilde y_3}\int_0^{\tilde y_4} \lambda e^{-\lambda y_1 y_2 y_3 y_4}\, \lambda e^{-\lambda y_1 (1-y_2) y_3 y_4}\, \lambda e^{-\lambda y_1 (1-y_3) y_4}\, \lambda e^{-\lambda y_1 (1-y_4)}\, |\det(J)|\, dy_1\, dy_2\, dy_3\, dy_4 \qquad (*)$$

$$\det(J) = (-1)(1-y_4) \underbrace{\det\begin{pmatrix} y_2 y_3 y_4 & y_1 y_2 y_4 & y_1 y_2 y_3 \\ -y_2 y_3 y_4 & y_1(1-y_2) y_4 & y_1(1-y_2) y_3 \\ 0 & -y_1 y_4 & y_1(1-y_3) \end{pmatrix}}_{A} - y_1 \underbrace{\det\begin{pmatrix} y_2 y_3 y_4 & y_1 y_3 y_4 & y_1 y_2 y_4 \\ (1-y_2) y_3 y_4 & -y_1 y_3 y_4 & y_1 (1-y_2) y_4 \\ (1-y_3) y_4 & 0 & -y_1 y_4 \end{pmatrix}}_{B}$$

$$A = y_1 y_4 \left(y_2 y_3 y_4 \cdot y_1(1-y_2) y_3 + y_2 y_3 y_4\, y_1 y_2 y_3 \right) + y_1 (1-y_3) \left(y_2 y_3 y_4\, y_1 (1-y_2) y_4 + y_1 y_3 y_4 \cdot y_1 y_2 y_4 \right)$$

$$= y_1^3 (1-y_2) y_3^2 y_4^2 + y_1^3 y_2 y_3^2 y_4^2 + y_1^3 (1-y_2) y_3 (1-y_3) y_4^2 + y_1^3 y_2 y_3 (1-y_3) y_4^2$$

$$= y_1^3 y_3^2 y_4^2 + y_1^3 y_3 (1-y_3) y_4^2 = y_1^3 y_3 y_4^2$$

$$B = (1-y_3) y_4 \left(y_1 y_3 y_4\, y_1 (1-y_2) y_4 + y_1 y_2 y_4\, y_1 y_2 y_4 \right) - y_1 y_4 \left(-y_2 y_3 y_4\, y_1 y_3 y_4 - (1-y_2) y_3 y_4\, y_1 y_3 y_4 \right)$$

$$= y_1^2 (1-y_2) y_3 (1-y_3) y_4^3 + y_1^2 y_2 y_3 (1-y_3) y_4^3 + y_1^2 y_2 y_3^2 y_4^3 + y_1^2 (1-y_2) y_3^2 y_4^3$$

$$= y_1^2 y_3 (1-y_3) y_4^3 + y_1^2 y_3^2 y_4^3 = y_1^2 y_3 y_4^3$$

$$\Rightarrow \det(J) = (-1)(1-y_4)(y_1^3 y_3 y_4^2) - y_1^3 y_3 y_4^3 = (-1)\, y_1^3 y_3 y_4^2$$

$*$. $y_1 y_2 y_3 y_4 + y_1 (1-y_2) y_3 y_4 + y_1 (1-y_3) y_4 + y_1 (1-y_4) =$

$y_1 y_3 y_4 + y_1 (1-y_3) y_4 + y_1 (1-y_4) =$

$y_1 y_4 + y_1 (1-y_4) = y_1$

$$\Rightarrow (*) = \int_0^{\tilde y_1}\int_0^{\tilde y_2}\int_0^{\tilde y_3}\int_0^{\tilde y_4} \lambda^4 e^{-\lambda y_1}\, y_1^3 y_3 y_4^2\, dy_1\, dy_2\, dy_3\, dy_4 .$$ Since the p.d.f. factorizes,

the random variables are independent.

Problem 4

(a) Let X be positive, to show: $\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X>t)\,dt$

$$\mathbb{E}[X] = \int_0^{+\infty} x \,\mathbb{P}(dx) = \int_0^{+\infty}\int_0^{x} dt\,\mathbb{P}(dx) = \int_0^{+\infty}\int_0^{+\infty} \mathbb{1}_{\{t<x\}}\,dt\,\mathbb{P}(dx) = \int_0^{+\infty}\int_0^{+\infty} \mathbb{1}_{\{X>t\}}\,\mathbb{P}(dx)\,dt = \int_0^{+\infty} \mathbb{P}(X>t)\,dt$$

(b.) Let $X \in \mathbb{N}$ be a random variable. To show: $\mathbb{E}[X] = \sum_{j=0}^{+\infty} \mathbb{P}(X>j)$

$$\mathbb{E}[X] = \sum_{K\geq 0} K\,\mathbb{P}(X=K) = \sum_{K\geq 1}\sum_{j=1}^{K} 1\,\mathbb{P}(X=K) = \sum_{K\geq 1}\sum_{j\geq 1} \mathbb{1}_{\{j\leq K\}}\,\mathbb{P}(X=K) = \sum_{j\geq 1}\sum_{K\geq 1} \mathbb{1}_{\{K\geq j\}}\,\mathbb{P}(X=K)$$

$$= \sum_{j\geq 1}\sum_{K\geq j} \mathbb{P}(X=K) = \sum_{j\geq 1} \mathbb{P}(X\geq j) = \sum_{j\geq 0} \mathbb{P}(X>j)$$

(c.) $$\mathbb{E}[X] = \mathbb{E}[X\,\mathbb{1}_{\{X\geq 0\}}] - \mathbb{E}[(-X)\,\mathbb{1}_{\{X<0\}}] = \int_0^{+\infty} \mathbb{P}(X>t)\,dt - \int_0^{+\infty} \mathbb{P}(-X>t)\,dt$$

$$= \int_0^{+\infty} (1-\mathbb{P}(X\leq t))\,dt - \int_0^{+\infty} \mathbb{P}(X<-t)\,dt \underset{t'=-t}{=} \int_0^{+\infty} (1-F_X(t))\,dt - \int_{-\infty}^{0} \mathbb{P}(X<t')\,dt' = \int_0^{+\infty} (1-F_X(t))\,dt - \int_{-\infty}^{0} F_X(t)\,dt.$$

(d.) When $X = Y^p$, Y being positive?

$$\mathbb{E}[Y^p] = \int_0^{+\infty} (1-\mathbb{P}(Y^p\leq t))\,dt = \int_0^{+\infty} (1-\mathbb{P}(Y\leq t^{1/p}))\,dt = \int_0^{+\infty} (1-F_Y(\sqrt[p]{t}))\,dt.$$